Answers to the Exam of Symmetry in Physics of April 10, 2015

Exercise 1

Consider an idealized beach ball with color pattern like in the picture below. If the north and south pole can be distinguished from each other, for instance by the color of the round patches, the symmetry group of the beach ball is a subgroup of $O(3)$ called C_{3v}.

(a) Identify all symmetry transformations that leave this beach ball invariant and divide them into conjugacy classes, using geometrical arguments.

> Define the $\hat{z}$ direction as the unit vector pointing from the middle of the ball to the north pole. Define the $\hat{x}$ direction as the unit vector pointing from the middle towards the center of one of the blue areas/bands. The $\hat{y}$ direction is defined such as to make a right-handed basis. The ball is invariant under the rotations around $\hat{z}$ over $0°, 120°, 240°$, which can be called e, c, c^2. The ball is invariant under reflections in the $\hat{x}$-$\hat{z}$ plane, and the two planes obtained from it by rotation around $\hat{z}$ over $120°$ and $240°$. The three reflections are called b_1, b_2, b_3. The operations b_i are thus clearly conjugated, being related to each other by a rotation. The two non-trivial rotations c and c^2 are conjugated because they can be related to each other by a reflection. e is in a class by itself. There are thus three conjugacy classes: (e), $(c) = (c^2)$, $(b_1) = (b_2) = (b_3)$.

(b) Give an identification between elements of C_{3v} and S_3 and argue that the two groups are isomorphic.

> S_3 is the group of permutations of 3 objects. Label the three blue areas/bands by 1,2,3. Then one establishes the bijective identification $c = (123)$, $c^2 = (132)$, $b_1 = (23)$, $b_2 = (13)$, $b_3 = (12)$ that shows that the elements of C_{3v} and S_3 follow the same group multiplication: $b_1 c = (23)(123) = (13) = b_2$ and $b_2 c = (13)(123) = (12) = b_3 = b_1 c^2$, which corresponds to the fact that a rotation followed by a reflection is again a reflection. Three-fold symmetries

in C_{3v} correspond to elements of order 3 in S_3, etc. In this way all elements of C_{3v} and S_3 can be expressed in terms of c and (for instance) b_1, such that the conditions $c^3 = (123)^3 = e$, $b_1^2 = (23)^2 = e$ and $(b_1 c)^2 = (13)^2 = e$ hold. Upon identification of $b = b_1$ this also establishes the isomorphism of both C_{3v} and S_3 with $D_3 = gp\{b, c\}$ with $c^3 = b^2 = (bc)^2 = e$.

(c) Write down the character table of C_{3v} and explain how the entries are obtained.

The character table is the same as that of D_3:

	(e)	(c)	(b)
$D^{(1)}$	1	1	1
$D^{(2)}$	1	1	-1
$D^{(3)}$	2	-1	0

For an explanation see section 2.6 of the lecture notes.

(d) Construct explicitly the three-dimensional vector representation D^V for the two transformations that generate C_{3v} and check the determinants.

$$D^V(c) = \begin{pmatrix} \cos(120°) & -\sin(120°) & 0 \\ \sin(120°) & \cos(120°) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D^V(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The determinants are +1 and -1, as expected for a rotation and reflection matrix in three dimensions, respectively.

(e) Decompose D^V of C_{3v} into irreps and use this to conclude whether this group allows for an invariant three-dimensional vector.

$\chi^V = (3, 0, 1)$, hence $D^V \sim D^{(1)} \oplus D^{(3)}$, which follows either by direct inspection or by calculating $\frac{1}{[g]} \sum_i k_i \chi^V(K_i) \chi^{(\nu)}(K_i)^* = a_\nu$. The decomposition includes the trivial rep, which shows that there is an invariant vector/direction. Clearly this corresponds to vectors pointing from south to north pole or vice versa.

(f) Determine the characters of the direct product representation $D^V \otimes D^V$ of C_{3v} and use them to determine the number of independent invariant tensors T^{ij} $(i, j = 1, 2, 3)$ (no need to construct them explicitly).

$\chi^{V\otimes V} = (9,0,1)$, which follows from squaring the characters of D^V. To find the number of independent invariant tensors, one has to determine a_1, which is the number of times the trivial irrep appears in the Clebsch-Gordan series of $D^V \otimes D^V$. Hence, one calculates $a_1 = \frac{1}{[g]}\sum_i k_i \chi^{V\otimes V}(K_i)\chi^{(1)}(K_i)^* = \frac{1}{6}(1\cdot 9\cdot 1 + 0 + 3\cdot 1\cdot 1) = 2$, to conclude that there are 2 invariant tensors (they are δ_{ij} and $\delta_{i3}\delta_{j3}$).

(g) Explain *in words* what changes if the north and south pole of the beach ball cannot be distinguished, i.e. explain what are the additional symmetries and whether an invariant three-dimensional vector is allowed or not.

Reflection in the $\hat{x}$-$\hat{y}$ plane is now an additional symmetry transformation. Adding it to the group C_{3v} doubles the number of elements. An invariant vector pointing in the $\hat{z}$ direction is now not allowed because after the new reflection it will point in the opposite direction, which means it has to be the zero vector if it needs to stay invariant.

Exercise 2

(a) Write down the defining representation of $SO(2)$.

The group is defined to be all 2×2 orthogonal matrices with determinant 1. Those matrices can be parametrized as:

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

which altogether for $0 \le \theta \le 2\pi$ form the defining representation.

(b) Write down the two-dimensional representation of $SO(2)$ obtained by its action on the vector

$$\begin{pmatrix} x+iy \\ x-iy \end{pmatrix}$$

Since $(x' \pm iy') = e^{\pm i\theta}(x \pm iy)$ under rotations, the rep is

$$D(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

(c) Show whether the two above representations are equivalent or not.

They are equivalent because they have the same trace: $2\cos\theta$. They only differ by a complex basis transformation that takes (x,y) into $(x+iy, x-iy)$.

(d) Explain whether the defining rep of $SO(2)$ is an irrep or not.

From the form in (b) one concludes that over the complex numbers the defining rep is reducible. Over the real numbers it is irreducible, which is not in conflict with Schur's lemma which refers to complex numbers.

(e) Show that the following matrices do not form a representation of $SO(2)$:

$$D(\theta) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \tag{1}$$

for a rotation over an angle θ.

Although it follows the expected multiplication $D(\theta_1 + \theta_2) = D(\theta_1)D(\theta_2)$, it has the property that $D(2\pi) = -D(0)$, i.e. the identity in $SO(2)$ is assigned to two distinct matrices: $\pm\mathbf{1}$. This means that it is not a homomorphism, because for instance $D(\pi)D(\pi) = -D(0)$, whereas in $SO(2)$ $D(\pi)D(\pi) = D(0)$. (In other words, (1) forms a projective rep of $SO(2)$.)

(f) Explain how wave functions transform under $SO(2)$ transformations.

Wave functions transform according to a unitary rep of $SO(2)$, to ensure that normalized wave functions remain normalized under the transformations. This answer suffices. In more detail: recall that the unitary irreps of $SO(2) \cong U(1)$ are $D^{(m)}(\theta) = (e^{i\theta m})$ for integer m (see for instance section 6.1 of the book). Hence, the wave functions will pick up a phase under $SO(2)$ transformations (possibly after a suitable basis tranformation). For $|l,m\rangle$ wave functions specifically, one can consider the $SO(2)$ transformations as rotations around the $\hat{z}$ axis, then

$$D^{(l)}_{m'm}(\theta) = \langle l,m'|U(R(\theta,\hat{z}))|l,m\rangle = \langle l,m'|e^{i\theta L_z/\hbar}|l,m\rangle = e^{i\theta m}\delta_{m'm}.$$

Therefore, the state $|l,m\rangle$ transforms according to $D^{(m)}$ irrespective of l.

Exercise 3

(a) Explain how a magnetic field transforms under rotations and reflections.

A magnetic field is an axial-vector, hence transforms as a vector under rotations, but transforms oppositely to a vector under reflections. Consider for example a closed current loop. The Biot-Savart law for the resulting magnetic field involves a vector product of two vectors yielding an axial-vector.

(b) Describe all symmetries of a three-dimensional space with a constant uniform magnetic field $\boldsymbol{B}$ pointing in the $\hat{z}$ direction (consider rotations, reflections, and translations).

> The system is invariant under rotations around the $\hat{z}$ direction, under reflections in the plane orthogonal to the $\hat{z}$ direction, and under translations in all directions. It is also invariant under space inversion or parity, which is a combination of reflection in the plane orthogonal to the $\hat{z}$ direction and a rotation over π, so this is an equivalent way of formulating the reflection symmetry of the system.

(c) Explain which components of the (linear) momentum and the orbital angular momentum should be conserved based on the symmetries obtained in part (b).

> Translation invariance in all directions implies conservation of all momentum components. Rotation invariance around the $\hat{z}$ axis implies conservation of the $\hat{z}$ component of the orbital angular momentum.

Consider a particle with charge q moving with velocity $\boldsymbol{v}$ in this constant uniform magnetic field. The particle will experience a Lorentz force

$$\boldsymbol{F} = q\,(\boldsymbol{v} \times \boldsymbol{B})\,. \tag{2}$$

(d) Show that the left- and right-hand side of the Lorentz force equation (2) transform in the same way under rotations and reflections.

> From the perspective of rotations the r.h.s. is a vector product of two vectors, hence a vector itself, like the l.h.s. Regarding reflections it is most convenient to consider reflection in the origin (space inversion or parity). Under parity the vector $\boldsymbol{v}$ picks up a minus sign, whereas the axial-vector $\boldsymbol{B}$ does not, so the r.h.s. picks up a minus sign, again like the vector $\boldsymbol{F}$ on the l.h.s.

(e) Ignoring cyclotron radiation emitted by the particle, its trajectory will be circular. Explain whether this trajectory is in accordance with the symmetries in part (b) of this question, and in case not, explain why not.

> Nonzero $\boldsymbol{F} = d\boldsymbol{p}/dt$ implies that momentum is not conserved. This may seem in contradiction with translation invariance, but the introduction of a particle into the system breaks the translation invariance. It introduces a special point in space, namely the place where the particle is at a specific moment. The circular trajectory has an angular momentum pointing in the $\hat{z}$ direction associated to it, which is conserved. The system is still in accordance with the rotational symmetry, only now around a specific point: the center of the circular orbit. The nonzero angular momentum is an axial vector, so allowed by the parity invariance of the system.